Magneto-Optical Tomography of Cosserat Flows

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Abstract A method of reconstruction of flows of the complex structure by means of the tomography technique based on the measurement of the change in the polarization of light passed through a birefringent medium, and Faraday effect is proposed. The reconstruction algorithm is based on the exponential Radon transform of an imaginary parameter for the cases of measurements in the presence and in the absence of a magnetic field.

1 Introduction

Magneto-optical tomography [1] is a nondestructive method for the three-dimensional stress analysis in transparent specimens. It is based on the measurement of the change in the polarization of light passed through a birefringent medium. The sample under study is placed in a strong constant magnetic field and illuminated in the direction parallel to this field. The plane of polarization of a light beam transmitted through the sample acquires an additional rotation, caused by the Faraday effect. The flow birefringence phenomenon in liquid is caused by presence of oblong molecules (polymers) or extended colloidal particles [2]. The particles at rest have both an isotropic and homogeneous distribution, and therefore the liquid is also optically isotropic. The flow causes orientation of particles along the current lines, and this fact creates the induced birefringence [3]. In case of the solution that contains polymeric chains, the flow besides orientation of molecules causes their extension and untwisting along the current lines [4]. Now polarizing methods are applied to estimation of qualitative behavior of flows [2]. The quantitative methods of flows study are basically limited by axisymmetric cases [4], [5].

The detailed study of possibilities of the standard approach of photoelasticity [6] has shown that the tomography technique gives possibility to restore the distribution of the velocity field of the flow partially. Only the axial component of velocity and a solenoidal component of the vector field can be reconstructed. The ultrasound tomography of flows based on measurement of time of a sound propagation [7] allows to determine an irrotational component of the vector field.

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It is necessary to notice that the experimental studies of Cosserat flows are carrying out in channels having the relatively small diameter (2-3 cm) that allows to place the used device into a magnetopolariscope [2] in practice.

From the mathematical point of view, the method of a parametrical magneto-optical tomography is based on use of the exponential Radon transform of an imaginary parameter applied to reconstruction of vector and tensor fields [1], [8]. As the proposed algorithm of reconstruction does not use the constitutive equations of the continua under study, in a sense, the offered technique is universal.

2 Exponential Radon Transformation of Vector and Tensor Fields

We define the imaginary-parameter exponential Radon transformation of a scalar function $f(x,y)$ by means of a ray path integral as

$$\begin{aligned} Er(s,\mathbf{k},\beta,f) &= \int f(s\cos\theta - t\sin\theta,\ s\sin\theta + t\cos\theta)e^{i\beta t}dt \\ &= \int f(s\mathbf{k}+t\mathbf{l})e^{i\beta t}dt = \breve{f}(s,\theta,\beta)\ . \end{aligned} \tag{1}$$

Here β is the transformation parameter, $\mathbf{k}=(\cos\theta,\sin\theta)$ and $\mathbf{l}=(-\sin\theta,\cos\theta)$ are mutually orthogonal unit vectors, located in a illuminated plane and directed along axes s and t correspondingly. The stationary coordinate system variables (x,y) are related through the moveable coordinate system variables (s,t) by the rotation transform

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}\ .$$

Derivation of the generalized Radon inversion formula

$$f(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \left\{ e^{-i\beta\mathbf{rl}} \int_{-\infty}^{\infty} \frac{\cosh[\beta(\mathbf{rk}-s)]}{\mathbf{rk}-s} \frac{\partial}{\partial s} \breve{f}(s,\theta,\beta)ds \right\} d\theta \tag{2}$$

was presented in [1].

According to Helmholtz theorem, a vector field $\mathbf{V}(x,y)$ can be uniquely presented in the form

$$\mathbf{V}(x,y) = \mathrm{grad}\tau + \mathrm{rot}\eta = \left(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\right)\tau + \left(\mathbf{e}_1\frac{\partial}{\partial y} - \mathbf{e}_2\frac{\partial}{\partial x}\right)\eta\ , \tag{3}$$

where the function τ is called the potential of an irrotational vector field and the function η is called the potential of a solenoidal vector field, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors directed along the stationary coordinate axes x,y .

The measurements in the linear vector tomography can be presented by ray path integral in the form of scalar (inner) product of vectors $\mathbf{p}$ and $\mathbf{V}$ as

$$Er(\mathbf{p}\cdot\mathbf{V}) = \int \mathbf{p}(s,\mathbf{k})\mathbf{V}(s\mathbf{k}+t\mathbf{l})e^{i\beta t}dt = \check{V}(s,\mathbf{k},\beta)\ . \tag{4}$$

Here $\mathbf{p}(s,\mathbf{k})$ is so called "probe" vector, which depends on the polar distance s, and on the direction of illumination $\mathbf{k}$. Using the representation of the vector field (3), the ray path integral (4) can be expressed via potentials τ and η in the form

$$\check{V}(s,\mathbf{k},\beta) = p_m k_m \left(\frac{\partial}{\partial s}\check{\tau} - i\beta\check{\eta}\right) - p_m l_m \left(\frac{\partial}{\partial s}\check{\eta} + i\beta\check{\tau}\right)\ . \tag{5}$$

In special case when the probe vector $\mathbf{p}=\mathbf{l}$ is directed along the ray, the ray path integral (5) reveals

$$Er(\mathbf{p}\cdot\mathbf{V}) = -\left(\frac{\partial}{\partial s}\check{\eta} + i\beta\check{\tau}\right)\ , \tag{6}$$

and this integral is called the longitudinal integral. In case of $\beta = 0$, the longitudinal integral depends only on the solenoidal component of the vector field.

In case when the probe vector $\mathbf{p}=\mathbf{k}$ is orthogonal to the ray (integrated photoelasticity), the ray path integral (5) leads to the formula

$$Er(\mathbf{p}\cdot\mathbf{V}) = \left(\frac{\partial}{\partial s}\check{\tau} - i\beta\check{\eta}\right)\ , \tag{7}$$

and is called the transverse integral. In case of $\beta = 0$, the transverse integral depends only on the derivative of the irrotational component of the vector field.

Thus, formulae (6) and (7) show that carrying out longitudinal or transverse measurements with $\beta = 0$ and $\beta \neq 0$, two-dimensional vector field can be reconstructed completely.

Similar the previous case, the invariant representation of the two-dimensional symmetric tensor σ_{ij} contains three scalar potentials G, N , F [1]

$$\begin{aligned}
\sigma_{xx} &= G + 2\frac{\partial^2}{\partial x\partial y}N + \frac{\partial^2}{\partial y^2}F\ ,\\
\sigma_{xy} &= -\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)N - \frac{\partial^2}{\partial x\partial y}F\ ,\\
\sigma_{yy} &= G - 2\frac{\partial^2}{\partial x\partial y}N + \frac{\partial^2}{\partial x^2}F\ .
\end{aligned} \tag{8}$$

The tensor stress and deformation field of plastic and viscous two-dimensional flows is determined by two-dimensional vector field of velocities $\sigma = \mathrm{def}(\mathbf{V})$. Expressing a vector $\mathbf{V}$ through potentials τ and η we obtain

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial}{\partial x}V_x = \frac{\partial^2}{\partial x^2}\tau + \frac{\partial^2}{\partial x\partial y}\eta\ ,\\
\sigma_{xy} &= \frac{1}{2}\left(\frac{\partial}{\partial x}V_y + \frac{\partial}{\partial y}V_x\right) = \frac{\partial^2}{\partial x\partial y}\tau - \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\eta\ ,
\end{aligned} \tag{9}$$

$$\sigma_{yy} = \frac{\partial}{\partial y} V_y = \frac{\partial^2}{\partial y^2}\tau - \frac{\partial^2}{\partial x \partial y}\eta \ .$$

The measurements in the linear tensor tomography can be presented by ray path integral in the form inner product of probe tensor $d(s,\mathbf{k})$ and measured tensor field σ as

$$\int d(s,\mathbf{k})\sigma(s\mathbf{k}+t\,\mathbf{l})e^{i\beta t}dt = Er[d(s,\mathbf{k})\cdot\sigma] \ . \tag{10}$$

In special case of tensor tomography when the probe tensor $d = \mathbf{k}\cdot\mathbf{k}$, the ray path integral (10) leads to the formula

$$Er(\mathbf{k}\cdot\mathbf{k}\cdot\sigma) = G - 2i\beta\frac{\partial}{\partial s}N - \beta^2 F \ , \tag{11}$$

and is called the transverse integral. Such kind of measurements are used in integrated photoelasticity during the studying of the stress field [1], for the measurements of the velocity field distribution in liquids [6], and plastic deformation and viscous flows [5]. The algorithm of reconstruction of residual stress in elastic and plastic media is presented in [1].

3 Optical Laws for Birefringent Media

The problem of light propagation in an inhomogeneous anisotropic medium is rather complicated. Usually, in the optical tomography this problem is simplified, and for description of distortion of the dielectric tensor by the physical fields is used the linear approximation of optical laws for birefringent media. For the simplest model of Cosserat continua considering the influence of the medium gyrotropy , the optical laws can be presented as:

- Neumann law for deformation

$$\varepsilon_{ij} = C_1\, e_{(ij)} + C_2\, \varepsilon_{ijm}\Phi_m + C_3\, \varepsilon_{ijm} V H_m \ , \tag{12}$$

- Neumann law for rate of strain

$$\varepsilon_{ij} = C_4\, \dot{e}_{(ij)} + C_5\, \varepsilon_{ijm}\dot{\Phi}_m + C_6\, \varepsilon_{ijm} V H_m \ , \tag{13}$$

- Maxwell law for stress

$$\varepsilon_{ij} = C_7\, \sigma^s_{ij} + C_8\, \sigma^a_{ij} + C_9\, \varepsilon_{ijm} V H_m \ , \tag{14}$$

- Filon-Jessop law for stress and deformation

$$\varepsilon_{ij} = C_{10}\, \sigma_{ij} + C_{11}\, e_{ij} + C_{12}\, \varepsilon_{ijm}\Phi_m + C_{13}\, \varepsilon_{ijm} V H_m \ , \tag{15}$$

where C_i - optical coefficients, $\varepsilon_{ij}, e_{ij}, \dot{e}_{ij}, \sigma_{ij}$ are the dielectric, deformation, strain-rate, and stress tensors, ε_{ijm} is Levi-Civita symbol, V is Verdet constant, H_m is the vector of magnetic field, and σ^s is the symmetric and σ^a is the antisymmetric part of the tensor σ . Here $e_{(ij)}$ is the symmetric part of the deformation tensor, and the rate-of-rotation vector Φ_m or vorticity

$$\Phi_m = \varepsilon_{ijm} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \tag{16}$$

considers the influence of gyrotropy of the media , and in the more general case includes rotation of particles. Here u_i is the displacement vector (deformation for plastic media). The influence of magnetic field (the Faraday effect) is included to the optical laws separately by additional term, and this effect can also be explained in frame of the Cosserat continuum theory [9].

The gyrotropy, or optical activity of the flows is studied yet not enough, and in the experiments with micropolar media the dielectric tensor is considered usually as the symmetric tensor (except for ferromagnetic mixtures and liquid crystals). Here and below, the antisymmetric part of the dielectric tensor is considered as caused by the Faraday effect only.

4 Ray Path Integrals in Weakly Birefringent Media

The polarization distortion in a weakly birefringent media is defined in quasi-isotropic approximation by equation [10]

$$(\mathbf{l}\,\nabla)\,\mathbf{E} = [A + B]\,\mathbf{E}\,, \tag{17}$$

where

$$\mathbf{E} = \begin{bmatrix} E_z \\ E_i\,k_i \end{bmatrix}, \qquad A = \frac{-i}{n_0} \begin{bmatrix} (\varepsilon^s_{zz} - \varepsilon^s_{ij} k_i k_j)/2 & \varepsilon^s_{zi} k_i \\ \varepsilon^s_{zi} k_i & -(\varepsilon^s_{zz} - \varepsilon^s_{ij} k_i k_j)/2 \end{bmatrix}, \tag{18}$$

$$B = \begin{bmatrix} 0 & \Upsilon \\ -\Upsilon & 0 \end{bmatrix}, \qquad \Upsilon = n_0^{-1} \varepsilon^a_{zi} k_i + V H_z\,. \tag{19}$$

Here E_i denote components of the Jones vector characterizing the polarization, n_0 is the refractive index, $\mathbf{l}$ is the unit vector located in an illuminated plane (xy) directed perpendicular to a direction of the ray, and $\mathbf{k}$ is the unit vector directed along the ray. The further consideration we will limit to a case of nongyrotropic medium (the antisymmetric part ε^a of the dielectric tensor is equal to zero), and for the constant value of the magnetic field, which direction coincides with that of light propagation.

The solution of equations (12) in linear approach relative to a symmetric part of coefficients it is possible to present in the form of two ray path integrals [8]

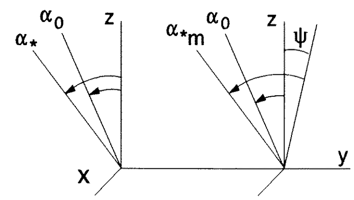

Fig. 1 Relationships between the characteristic angles

$$\gamma \cos(\alpha_* + \alpha_0 + \psi) = C\left\{\int \frac{1}{2}[\varepsilon_{zz} - \varepsilon_{ij} k_i k_j] \cos(2\beta t) dt - \int \varepsilon_{zi} k_i \sin(2\beta t) dt\right\} , \quad (20)$$

$$\gamma \sin(\alpha_* + \alpha_0 + \psi) = C\left\{\int \frac{1}{2}[\varepsilon_{zz} - \varepsilon_{ij} k_i k_j] \sin(2\beta t) dt + \int \varepsilon_{zi} k_i \cos(2\beta t) dt\right\} , \quad (21)$$

which values are defined experimentally, as a result of illumination. Here γ is the characteristic phase difference, α_0 is the initial characteristic direction measured in the initial coordinate system, $\alpha_{*m} = \alpha_* + \psi$ and α_* are the secondary characteristic directions (the angles α_{*m} and α_* are measured in the movable and initial coordinate systems), and $\alpha_m = \alpha_{*m} - \alpha_0 = \alpha + \psi$ and $\alpha = \alpha_* - \alpha_0$ are the characteristic angles measured in accordance with figure 3.

The ray path integral parameters α_0 and α_* are measured directly during experiment, and $\psi = V\int H dy = VHy = \beta y$ is the angle of rotation of the plane of polarization, which is caused by the Faraday effect.

Both ray path integrals (20) and (21) can be presented in the form of two transverse integrals - the first as a function of two-dimensional tensor field

$$H_1(s, \mathbf{k}, \beta) = \int (\varepsilon_{zz} - \varepsilon_{ij} k_i k_j) e^{i\beta t} dt , \quad (22)$$

and the second one as a function of two-dimensional vector field

$$H_2(s, \mathbf{k}, \beta) = \int \varepsilon_{zj} k_j e^{i\beta t} dt . \quad (23)$$

Let us remind, that both integrals differ from usually used in standard measurements [3], [4], [10] by presence of exponential multiplier in integrand term.

5 Tomography of Viscous Flows

The mathematical problem of determination of a velocity field $\mathbf{V}$ of viscous flow is formulated in the form of incompressibility condition div $\mathbf{V} = 0$, and boundary condition $\mathbf{V} = 0$ on the wall.

The further consideration we will limit to a case of linear dependence between the dielectric tensor ε_{ij} and the strain-rate tensor $\dot{e}_{ij}$ (Neumann law). The components of the strain-rate tensor

$$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{24}$$

are equal to deformation of vector velocity of flow $\mathbf{V}(x,y)$. Using representation (3) of the vector field in the form of an irrotational τ and a solenoidal η fields, the velocity field can be written as

$$V_x = \frac{\partial \tau}{\partial x} + \frac{\partial \eta}{\partial y}\ , \tag{25}$$

$$V_y = \frac{\partial \tau}{\partial y} - \frac{\partial \eta}{\partial x}\ . \tag{26}$$

On the basis of ray path integrals (22), (23), and using relationships (7), (9), (11), the tomography measurements in the absence of a the magnetic field ($\beta = 0$) allow to restore only the axial component of velocity V_z and potential τ [6].

To restore the solenoidal part of the field, we write the ray path integral (22) in a more adjustable form using the incompressibility condition div $\mathbf{V} = 0$, and relations (24), (25), (26)

$$H_1 = C\int[\dot{e}_{zz} - \dot{e}_{xx}]e^{i\beta y}dy =$$

$$C\int\left(2\frac{\partial V_z}{\partial z} + \frac{\partial V_y}{\partial y}\right)e^{i\beta y}dy = \{\mathbf{K}\} + C\int\frac{\partial^2\eta}{\partial x\partial y}e^{i\beta y}dy\ . \tag{27}$$

The left side of this expression H_1 is the known value obtained from measurements, and the term $\{\mathbf{K}\}$ is the part of the field, which can be determined beforehand, in the absence of magnetic field ($\beta = 0$). Thus, using the Radon inverse formula (2) provides an opportunity to determine solenoidal part of the field η, and also other components of a field V_x and V_y, defined by the equations (25), (26). Till now, only the Doppler tomography [7] was used for reconstruction of a solenoidal component of a vector field.

6 Use of Dispersion for Fields Reconstruction

In previous sections on the basis of Neumann and Maxwell optical laws the problems of tomographic reconstruction of stress and velocity fields of viscous flows have been considered. The algorithm of reconstructing of plastic deformation in axisymmetric case has been considered in [5]. The method of reconstruction of plastic deformations for a three-dimensional case is similar to a method of reconstruction of plastic flows. Herewith the incompressibility condition and zero boundary conditions also are used.

The case of the Filon - Jessop law is the subject to special consideration. The technique of tomographic reconstruction of flows in a two-dimensional case is based on the dependence of coefficients C_{10} and C_{11} in relation (15) on the frequency of light (dispersion). Measuring optical parameters at different frequencies, we can obtain the linear system of equations, from which the ray path integrals for deformation and stress can be determined separately [1]. Thus, using the procedure mentioned above, it is possible also to restore the deformation and stress of the medium that is described by the Filon - Jessop law as well..

7 Conclusions

Algorithms of tomographic reconstruction of stress and deformation of plastic and viscous flows are presented. It is necessary to notice that for reconstruction of all aforementioned characteristics of the flow, the constitutive equations of the continua (medium with nonlinear mechanical properties) were not used. The essential role in reconstruction is played by the linearity of optical laws. In case of weak nonlinearity of optical laws, the presented algorithms can be used as a basis for reconstruction of corresponding fields by Newton-Kantorovich's method.

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